Appendix A Proofs and Auxiliary Lemmas

Proof of Theorem 15. Let π denote the witnessing proof of ϕ. Let π' be the proof $\sigma(\pi)$ replacing each leaf $\sigma(\psi_i)$ with $\psi_i \in H$ by the corresponding well-supported proof π_i of $\frac{K}{\sigma(\psi_i)}$. We claim that π' witnesses $\frac{K}{\sigma(\phi)}$. We show this by induction on π.

In the case of a hypothesis, then π' is some π_i and ϕ is ψ_i, and we are done.

For the positive step, then π concludes with an application of a deduction rule under substitution τ. Then π' concludes with the same deduction rule under substitution $\sigma \circ \tau$.

For the negative case, suppose $\phi = s \stackrel{l}{\nrightarrow}$ so the root of π' is $\frac{\sigma(K)}{\sigma(s) \stackrel{l}{\nrightarrow}}$. Let τ be a substitution, ϕ' deny $\tau(\sigma(s) \stackrel{l}{\nrightarrow}) = \tau \circ \sigma(s \stackrel{l}{\nrightarrow})$ and π'' conclude ϕ'. Since π is a well-supported proof, there exists $\psi \in K$ with ψ' denying $\tau \circ \sigma(\psi)$ occurring in π''. But then $\sigma(\psi) \in \sigma(K)$, ψ' denies $\tau(\sigma(\psi))$ and ψ' occurs in π'', as required. □

Proof of Corollary 16. i) By the construction in the proof of Theorem 15, if H is empty then $\pi' = \sigma(\pi)$ is a well-supported proof of $\overline{\sigma(\phi)}$. ii) Suppose ϕ is closed. Let σ map each variable to the source of ϕ. By (i), $\sigma(\pi)$ is a well-supported proof of $\overline{\sigma(\phi)} = \overline{\phi}$. But $\sigma(\pi)$ is a closed proof, as required. □

Proof of Theorem 17. We first show that the set of (provable ruloid) derivations remains intact under instantiation. It trivially holds that if ϕ is a derivation from H w.r.t. T, it is also a derivation w.r.t. T'. It thus remains to check the implication in the reverse direction.We proceed by induction on the depth of the derivation for $\frac{H}{\phi}$.

If the derivation appeals to a hypothesis in H, then the provable ruloid is clearly valid in T as well. Otherwise, ϕ must be positive and the set K of formulae are placed above ϕ is such that $\frac{K}{\phi}$ is the result of applying a substitution σ to a deduction rule d in T'. If d is in T, then the thesis follows from the induction hypothesis. Otherwise if d is T' but not in T, it is the result of applying a substitution σ' to a deduction rule d' from T. Hence, by applying $\sigma \circ \sigma'$ to d', one can obtain the instance $\frac{K}{\phi}$. By induction, the proof subtrees for the ruloids rooted in members of K can be reconstructed using deduction rules of T.

We must now show the same of well-supported proofs, i.e. a well-supported proof π for $\frac{H}{\phi}$ in T' is a well-supported proof in T, and *vice versa.* We proceed by induction on the proof. The case for hypotheses and instances of deduction rules follow exactly as in the case for provable ruloids. For the negative case, let $\frac{H}{s \stackrel{l}{\nrightarrow}}$ be the root of derivation π in T. We wish to show that it is also a derivation in T'. To do this, we show that for each π' witnessing provable ruloid $\frac{K}{\sigma(s) \stackrel{l}{\longrightarrow} s'}$ in T', a formula occurring in π' denies $\sigma(h)$ for $h \in H$. Since any such provable ruloid derivation is also one in T and π is a well-supported proof in T, we know this is the case. The negative induction step from T' to T follows similarly. □

Lemma 36 *Let $T_0 \uplus T_1$ be a disjoint extension of T_0. Let s be a term in the signature of $T_0 \uplus T_1$, and t, r be terms in the signature of T_0. Let σ, τ be substitutions such that $\sigma(r) = \tau(t) = s$. Then there exists substitutions $\hat{\sigma}, \hat{\tau} \in T_0$ and $\rho \in T_0 \uplus T_1$, such that $\sigma = \rho \circ \hat{\sigma}$, $\tau = \rho \circ \hat{\tau}$ and $\hat{\sigma}(r) = \hat{\tau}(t)$.*

Proof. For a term s, define $|s|$ by induction: $|x| = 0$, $|f(s_1, \ldots, s_n)| = 1 + |s_1| + \ldots + |s_n|$. For terms s, t define $\mathsf{d}(s,t)$ as follows: $\mathsf{d}(x,t) = \mathsf{d}(t,x) = |t|$, $\mathsf{d}(f(s_1,\ldots,s_n), f(t_1,\ldots,t_n)) = \mathsf{d}(s_1,t_1) + \ldots + \mathsf{d}(s_n,t_n)$ and $\mathsf{d}(f(s_1,\ldots,s_n), g(t_1,\ldots,t_m)) = \infty$ for $f \neq g$.

We proceed by induction on $\mathsf{d}(r,t)$. If $\mathsf{d}(r,t) = 0$, then r and t are the same up to renaming of variables. Since $\sigma(r) = \tau(t)$, there is a total surjective relation $R : \mathsf{vars}(r) \leftrightarrow \mathsf{vars}(t)$ such that xRy implies $\sigma(x) = \tau(y)$. Define an equivalence relation on $\mathsf{vars}(r)$ by $x_1 \sim_r x_n$ if $x_1 R y_1 R^{-1} x_2 R y_2 \ldots R y_{n-1} R^{-1} x_n$ where $yR^{-1}x$ if and only if xRy, and similarly for $\mathsf{vars}(t)$. Then $x \sim_r x'$ implies $\sigma(x) = \sigma(x')$, and similar for t. Let $[x]_r$ denote the least y with $y \sim_r x$, and similar for t. Let $f : \mathsf{vars}(r) \to \mathsf{vars}(t)$ be defined by $f(x) = [y]_t$ for xRy. Then $\tau(f(x)) = \sigma(x)$. Let $g(x) = [x]_t$. Then $f(r) = g(t)$. Let $\hat{\tau}$ send $x \in \mathsf{var}(t)$ to $\mathsf{in}_2([x]_t)$ and $x \notin \mathsf{var}(t)$ to $\mathsf{in}_2(x)$. Let $\hat{\sigma}$ send $x \in \mathsf{vars}(r)$ to $\mathsf{in}_2(f(x))$ and x to $\mathsf{in}_1(x)$ otherwise. Then $\hat{\sigma}(r) = \mathsf{in}_2(f(r)) = \mathsf{in}_2(g(t)) = \hat{\tau}(t)$. Let $\rho = [\sigma,\tau]$. Then $\rho \circ \hat{\tau} = \tau$: for $x \in \mathsf{vars}(t)$, $\rho \circ \hat{\tau}(x) = [\sigma,\tau] \circ \mathsf{in}_2([x]_t) = \tau([x]_t) = \tau(x)$; otherwise, $\rho \circ \hat{\tau}(x) = [\sigma,\tau] \circ \mathsf{in}_2(x) = \tau(x)$. Finally, $\rho \circ \hat{\sigma} = \sigma$: for $x \in \mathsf{vars}(r)$, $\rho \circ \hat{\sigma}(x) = [\sigma,\tau] \circ \mathsf{in}_2(f(x)) = \tau(f(x)) = \sigma(x)$; otherwise $\rho \circ \hat{\sigma}(x) = [\sigma,\tau] \circ \mathsf{in}_1(x) = \sigma(x)$.

The case $\mathsf{d}(t,r) = \infty$ is impossible, since $\sigma(r) = \tau(t)$.

If $0 < \mathsf{d}(t,r) < \infty$, then (without loss of generality) there must be a position within r that is a variable x while the corresponding position within t is a compound term $f(t_1,\ldots,t_n)$ where f is a symbol from T_0, with $\sigma(x) = \tau(f(t_1,\ldots,t_n))$. Let r' be $\mathsf{in}_1[x \mapsto f(\mathsf{in}_2(x_1),\ldots,\mathsf{in}_2(x_n))](r)$ where $x_1,\ldots,x_n$ are distinct variables. Let $\sigma' = [\sigma,\kappa]$ where κ sends x_i to $\tau(t_i)$. Then $\sigma'(r') = s$. Now $\mathsf{d}(t,r') < \mathsf{d}(t,r)$ and so by inductive hypothesis there exists $\hat{\sigma'}, \hat{\tau} \in T_0$ and ρ such that $\rho \circ \hat{\tau} = \tau$, $\rho \circ \hat{\sigma'} = \sigma'$ and $\hat{\tau}(t) = \hat{\sigma'}(r')$. Now, $r' = \mu(r)$ where $\mu = \mathsf{in}_1[x \mapsto f(\mathsf{in}_2(x_1),\ldots,\mathsf{in}_2(x_n))]$ and $\sigma = \sigma' \circ \mu = \rho \circ \hat{\sigma'} \circ \mu$. Set $\hat{\sigma} = \hat{\sigma'} \circ \mu$. Then $\rho \circ \hat{\tau} = \tau$, $\rho \circ \hat{\sigma} = \sigma$ and $\hat{\sigma}(r) = \hat{\sigma'} \circ \mu(r) = \hat{\sigma'}(r') = \hat{\tau}(t)$, as required. □

Proof of Lemma 22. If ψ, ϕ and ω are negative, we may apply Lemma 36 to the respective sources and we are done. If they are positive, we may proceed just as in the proof of Lemma 36, treating $\xrightarrow{l}$ as a binary function symbol. □

Proof of Theorem 28. Our inductive hypotheses require something stronger than the stated theorem: we require that only the source and label of ϕ are in T_0.

Derivations: We show that π is a derivation in T_0, proceeding by induction. Let s be the source of ϕ and l the label of ϕ.

If π just appeals to a hypothesis, then ϕ must itself appear in H and be in T_0, and so π is a derivation in T_0, as required.

Otherwise, π must appeal to some deduction rule d under substitution σ. Let d be of the form $\frac{\{\rho_i : i \in I\}}{\rho}$. Let $\{\pi_i : i \in I\}$ be the set of formulae in the proof-tree placed immediately above each $\phi_i = \sigma(\rho_i)$. Let s_i be the source of ϕ_i, l_i the label of ϕ_i and r_i the source of ρ_i. Let $\phi = s \xrightarrow{l} s'$ and $\rho = r \xrightarrow{l} r'$.

Note that deduction rule d must be in T_0, since otherwise the head symbol of d is not in T_0, which is impossible as $\sigma(r) = s$ in T_0.

We next show that, for each i: s_i and l_i are in T_0 and π_i witnesses the provable ruloid $\frac{\Gamma'}{\phi_i}$ in T_0. Thus, the target of ϕ_i (if it has one) is in T_0. For each variable x in d, define $\delta(x)$ to be the ordinal witnessing the least number of steps required to show that x is source-dependent according to the inductive definition. For each i, define δ_i to be the maximal $\delta(x)$ such that x appears in r_i. We show the above claim by induction on δ_i.

Let $V_i = \{x \in \mathsf{vars}(\rho_j) : \delta_j < \delta_i\} \cup \mathsf{vars}(r)$. Then $\mathsf{vars}(r_i) \subseteq V_i$. By inductive hypothesis, for $\delta_j < \delta_i$, $\phi_j \in T$. Since $\phi_j = \sigma(\rho_j) \in T_0$, it follows that $\sigma(x) \in T_0$ for each x in such a ρ_j. Similarly, since $s = \sigma(r) \in T_0$, for all $x \in \mathsf{vars}(r)$, $\sigma(x) \in T_0$. So for all $x \in V_i$, $\sigma(x) \in T_0$. Since $r_i \in T_0$ and $\mathsf{vars}(r_i) \subseteq V_i$, $\phi_i = \sigma(r_i) \in T$. Further, l_i is in T, since it occurs in T_0-rule d. We may then apply the (outer) inductive hypothesis to see that π_i is a derivation in T_0 and so the target of ϕ_i is in T_0.

Finally, $\mathsf{vars}(r') \subseteq \mathsf{vars}(r) \cup \{\mathsf{vars}(\rho_i) : i \in I\}$. Any such variable is mapped to a T_0-term by σ. Then since $r' \in T_0$, so is $s' = \sigma(r')$.

The proof π applies deduction rule d (in T_0) to the derivations π_i (each in T_0) to derive transition ϕ (which is in T_0). We can conclude that π itself is in T_0.

Well-supported proofs: Let π be a well-supported proof of $\frac{\Gamma}{\phi}$, we proceed by induction on π.

If π appeals to a hypothesis or a deduction rule, we can proceed exactly as in the provable ruloid case.

If ϕ is negative $s \overset{l}{\nrightarrow}$ and the set $\{\phi_i \mid i \in I\}$ are immediately placed above ϕ, each with a subproof π_i, then we must show that for each provable ruloid derivation π' concluding $\sigma(s) \xrightarrow{l} s'$ in T_0, there is a formula in π' denying some $\sigma(\phi_i)$. Each such provable ruloid is also valid in $T_0 \uplus T_1$ by applying Theorem 26. Since π is a well-supported proof in $T_0 \uplus T_1$ of $s \overset{l}{\nrightarrow}$, there exists a formula occurring in π' denying some $\sigma(\phi_i)$, as required. □

Proof of Lemma 31. We remove each violating instance, one at a time. If $s \overset{l}{\nrightarrow}$ occurs above $r \overset{m}{\nrightarrow}$ and s is not a variable, then $s \overset{l}{\nrightarrow}$ cannot be a hypothesis. We then replace

$$\frac{\dfrac{\{\phi_i : i \in I\}}{s \overset{l}{\nrightarrow}} \qquad \{\psi_j : j \in J\}}{r \overset{m}{\nrightarrow}}$$

by $\frac{\{\phi_i : i \in I\} \cup \{\psi_j : j \in J\}}{r \overset{m}{\nrightarrow}}$ (this preserves closedness). To see that this is still a well-supported proof, let π witness a provable ruloid concluding $\sigma(r) \xrightarrow{m} r'$. Then there is a formula in π which denies either some $\sigma(\psi_j)$ or $\sigma(s \overset{l}{\nrightarrow})$. In the former case, we are done. Otherwise, $\sigma(s) \xrightarrow{l} s'$ occurs in π, and we may consider the subderivation π' concluding $\sigma(s) \xrightarrow{l} s'$, also a provable ruloid derivation. Since its conclusion denies $\sigma(s \overset{l}{\nrightarrow})$, there is a formula occurring in π' denying $\sigma(\phi_i)$. But this formula then also occurs in π, and so the generated well-supported proof is valid. $\square$

Proof of Proposition 34. Let π be the witnessing derivation, we show that π is also witnesses the provable ruloid in *closed*(T). We proceed by induction on ϕ. If ϕ appeals to a hypothesis then since ϕ is closed, this is also valid in $\mathsf{closed}(T)$. Otherwise, π must appeal to some derivation rule d under substitution σ. Let d conclude ρ from premises $\{\rho_i : i \in I\}$. Let $\{\pi_i : i \in I\}$ be the immediate children of π, each proving ϕ_i. Let s_i be the source of ϕ_i, l_i the label of ϕ_i and r_i the source of ρ_i. Let $\phi = s \xrightarrow{l} s'$ and $\rho = r \xrightarrow{l} r'$.

We show that for each i premise, s_i is closed and π_i is a provable ruloid in *closed*(T). We proceed by induction on δ_i. The variables of each r_i appear r'_j for $j < i$ and r. Since $s = \sigma(r)$ and $s'_j = \sigma(r'_j)$ are each closed, $\sigma(x)$ is a closed term for each variable in r_i. Resultantly, $s_i = \sigma(r_i)$ is a closed term. We may thus apply the (outer) inductive hypothesis to see that π_i is a proof in *closed*(T), and s'_i is closed. Finally, as all variables in r' appear in some r'_j or r, we see that $s' = \sigma(r')$ is also a closed term. Any instance of a rule in T applied to closed terms is also an instance of a rule in *closed*(T), and so π is a valid proof in *closed*(T). $\square$