Modular Semantics for Open Transition Rules with Negative Premises

Martin Churchill, Peter D. Mosses, Mohammad Reza Mousavi
Swansea University Halmstad University

Queen Mary University of London
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Structural Operational Semantics and Negative Premises

- Structural Operational Semantics specifies a transition (evaluation) relation \( \xrightarrow{\ell} \) via *inductive rules*.

- Sometimes, authors of process algebras like to use negative premises. E.g.:

  \[
  \begin{array}{c}
  x \xrightarrow{\ell} x' \\
  \hline
  x; y \xrightarrow{\ell} x'; y
  \end{array}
  \]

  \[
  \begin{array}{c}
  \{x \xrightarrow{\ell}\} y \xrightarrow{m} y' \\
  \hline
  x; y \xrightarrow{m} y'
  \end{array}
  \]

- Sometimes negative premises are needed, e.g. certain priority operators inexpressible using just positive premises [Aceto and Ingólfsdóttir(2008)].
Semantics of Systems with Negative Premises?

- No longer a simple inductive definition of provable transitions.
- Potential pitfalls, e.g. rules like $a \not\rightarrow b$ implies $a \not\rightarrow b$.
- Various approaches, that of well-supported proofs is a popular & powerful notion [Glabbeek(2004)].
- Is incomplete for pathological examples like that above:
  - neither $a \not\leftrightarrow$ nor $a \not\rightarrow b$ are derivable
  - by restricting attention to complete specifications, one achieves a 2-valued TSS
Towards open formulae

- Well-supported proof only works for closed formulas
  - Asserting provability of $s \xrightarrow{I} s'$ or $s \nrightarrow I$ for closed $s,s'$.
- We wish to extend the notion to open formulae, with hypotheses are variables. e.g.

\[
\frac{\{x \xrightarrow{I}\} I \quad \{y \xrightarrow{I}\} I \quad z \xrightarrow{m} z'}{(x; y); z \xrightarrow{m} z'}
\]
Towards open formulae

- Well-supported proof only works for closed formulas
  - Asserting provability of $s \xrightarrow{I} s'$ or $s \nrightarrow$ for closed $s,s'$.
- We wish to extend the notion to open formulae, with hypotheses are variables. e.g.

$$
\begin{array}{c}
\{ x \nrightarrow \} \downarrow \\
\{ y \nrightarrow \} \downarrow \\
z \xrightarrow{m} z' \\
\end{array}
\Rightarrow
(x; y) ; z \xrightarrow{m} z'
$$

- Why?
  - To support (open) operational laws via (fh-)bisimulation which remain valid under disjoint extensions [Mosses et al.(2010)Mosses, Mousavi, and Reniers]
  - e.g. $(x; y) ; z \sim x ; (y ; z)$
A notion of **well-supported proof** for open transition rules satisfying various desirable properties:

- Consistency (\(s \vdash s'\) and \(s \nvdash\) can’t both be provable)
- Instantiation closure (if \(\bar{s}\) is provable then so is \(\sigma(s)\))
- Agrees with original notion on closed terms
- Modularity (under disjoint extensions, old proofs remain valid)
- Conservativity (under disjoint extensions, no new proofs of old formulae)
<table>
<thead>
<tr>
<th>Introduction</th>
<th>Well-supported Proofs for Closed Formulae</th>
</tr>
</thead>
<tbody>
<tr>
<td>Results</td>
<td>Well-Supported Proofs for Open Transition Rules</td>
</tr>
<tr>
<td>Conclusions</td>
<td></td>
</tr>
</tbody>
</table>
Basic Notions

Transition System Specifications have:

- A signature $\Sigma$ and set of labels $L$.
- Formulas $\phi$ are of the form $s \xrightarrow{l} s'$ or $s \xrightarrow{l}$ where $s, s'$ are $\Sigma$-terms and $l \in L$.
  - $s \xrightarrow{l} s'$ denies $s \xrightarrow{l}$ and vice-versa.
- A set of deduction rules $\frac{H}{s \xrightarrow{l} s'}$ over such formulas.

A derivation of a transition rule $\frac{H}{\phi}$ is an inductive proof using rules in $D$ with open leaves/hypotheses (possibly negative) in $H$. 
A ground well-supported proof of $\phi$ is a upwardly branching tree labelled by closed formulae and rooted at $\phi$, where:

- Positive steps $\frac{K}{s \rightarrow s'}$ are instances of deduction rules

- For negative steps $\frac{K}{s \nrightarrow}$, it must be the case that:

Each derivation of $\frac{N}{s \rightarrow s'}$, ($N$ negative) contains some formula which denies a formula in $K$

Negative steps work by refuting each possible derivation.
As we will see, the above definition doesn't work for open formulae / transition rules.

An alternative is closed-instance semantics: $\phi$ holds for open $\phi$ if all closed instantiations $\sigma(\phi)$ holds.

But this fails to be modular:

1. In a base system with single rule $\frac{x \xrightarrow{b} x}{f(x) \xrightarrow{a} x}$, $f(x) \xrightarrow{a} x$ holds.
2. But disjointly adding $1 \xrightarrow{b} 1$ invalidates the formula.
Towards open formulae

Example

Consider a TSS with deduction rules \( f(x) \overset{a}{\rightarrow} \), \( f(0) \overset{a}{\rightarrow} 0 \). Then:

\[ \begin{align*}
\rightarrow & \quad f(1) \overset{a}{\not\rightarrow} \quad \text{and} \quad g(1) \overset{a}{\rightarrow} 1 \quad \text{have well-supported proofs.} \\
\rightarrow & \quad \text{The derivation } f(0) \overset{a}{\rightarrow} 0 \quad \text{ensures that neither } f(0) \overset{a}{\not\rightarrow} \quad \text{nor } \\
\rightarrow & \quad g(0) \overset{a}{\rightarrow} 0 \quad \text{are provable.} \\
\rightarrow & \quad f(x) \overset{a}{\rightarrow} \quad \text{is provable... shouldn’t be, due to the derivation} \\
\rightarrow & \quad f(0) \overset{a}{\rightarrow} 0 \quad \text{which denies an instance of } f(x) \overset{a}{\rightarrow}. \end{align*} \]
Towards open formulae

Example

Consider a TSS with deduction rules \( \frac{f(x) \xrightarrow{a} \not\rightarrow}{g(x) \xrightarrow{a} x} \), \( f(0) \xrightarrow{a} 0 \). Then:

- \( f(1) \xrightarrow{a} \) and \( g(1) \xrightarrow{a} 1 \) have well-supported proofs.
- The derivation \( f(0) \xrightarrow{a} 0 \) ensures that neither \( f(0) \xrightarrow{a} \) nor \( g(0) \xrightarrow{a} 0 \) are provable.
- \( f(x) \xrightarrow{a} \) is provable... shouldn’t be, due to the derivation \( f(0) \xrightarrow{a} 0 \) which denies an instance of \( f(x) \xrightarrow{a} \).

\( \Rightarrow \) We must consider counterexamples up to substitution: otherwise, \( g(x) \xrightarrow{a} x \) provable, but \( g(0) \xrightarrow{a} 0 \) unprovable.
Well-supported proofs for open formulas

We next adapt the notion of well-supported proof to open transition rules \( \frac{H}{\phi} \) where \( H \) is a context:

▶ \( H \) gives assumptions on variables \((x \xrightarrow{I} s, x \xrightarrow{I} )\).
Well-supported proofs for open formulas

A well-supported proof of \( \frac{H}{\phi} \) is a upwardly branching tree labelled by formulae and rooted at \( \phi \), where:

- Leaves are in \( H \)
- Positive steps \( \frac{K}{l} \) are instances of deduction rules
- For negative steps \( \frac{K}{l} \), it must be the case that:

Each derivation of \( \frac{C}{\sigma(s)} \), \( (C \text{ negative } + \text{ vars}) \)

contains a formula denying \( \sigma(k) \) for some \( k \in K \)

(Differences from closed version: 
\( H \) hypotheses, substitutive counter examples.)
Basic Results
Theorem (Closure under Instantiating Formulae)

Suppose $\left\{ \psi_i : i \in I \right\}$ has a well-supported proof.

Let $\sigma$ be a substitution so each $\frac{K}{\sigma(\psi_i)}$ has a well-supported proof.

Then $\frac{K}{\sigma(\phi)}$ has a well-supported proof.
**Theorem (Closure under Instantiating Formulae)**

Suppose \( \frac{\psi_i : i \in I}{\phi} \) has a well-supported proof. Let \( \sigma \) be a substitution so each \( \frac{K}{\sigma(\psi_i)} \) has a well-supported proof. Then \( \frac{K}{\sigma(\phi)} \) has a well-supported proof.

Proof: Substitution + pasting of proof trees.
Theorem (Consistency)

In any TSS, it can't be the case that \( s \xrightarrow{I} s' \) and \( s \xrightarrow{I} \) both have well-supported proofs.
Consistency

Theorem (Consistency)

In any TSS, it can’t be the case that $s \rightarrow s'$ and $s \not\rightarrow$ both have well-supported proofs.

Proof (contradiction): assume minimal proofs of contradicting formulae. use “root derivation” of positive part with negative part to find smaller contradicting proofs.
Consistency

**Theorem (Consistency)**

*In any TSS, it can't be the case that \( s \rightarrow s' \) and \( s \nrightarrow \) both have well-supported proofs.*

Proof (contradiction): assume minimal proofs of contradicting formulae. use “root derivation” of positive part with negative part to find smaller contradicting proofs.

**Generalisation:** Some consistency assumptions on \( H \Rightarrow \) can’t prove both \( \frac{H}{s \rightarrow s'} \) and \( \frac{H}{s \nrightarrow} \)
Modularity
A disjoint extension of a TSS is:

- An extension of the signature $\Sigma$ with new symbols $\Sigma'$ and labels
- An extension of $D$ with new rules $D'$, each of which is of the form $S$ for $f \in D'$.

$$f(s_1, \ldots, s_n) \rightarrow t$$
A *disjoint extension* of a TSS is:

- An extension of the signature $\Sigma$ with new symbols $\Sigma'$ and labels.
- An extension of $D$ with new rules $D'$, each of which is of the form $S_f(s_1, \ldots, s_n) \rightarrow t$ for $f \in D'$.

Important property: If $\pi$ is a well-supported proof of $\frac{H}{\phi}$ in $T$, then remains so in $T \cup T_1$.

For positive steps $\frac{K}{s \rightarrow s'}$, simple.
Modularity for Negative Steps

For negative steps we need:

\[
\frac{K}{s \not\in} \quad \text{is valid in } T_0 \Rightarrow \text{valid in } T_0 \cup T_1.
\]

i.e. each counterexample proving \( \sigma(s) \rightarrow s' \) must be denied for \( \sigma \in T_0 \cup T_1 \)
Modularity for Negative Steps

For negative steps we need:

$$\frac{K}{s \not\rightarrow}$$

is valid in $T_0 \Rightarrow$ valid in $T_0 \uplus T_1$.

i.e. each counterexample proving $\frac{C}{\sigma(s) \rightarrow s'}$ must be denied for $\sigma \in T_0 \uplus T_1$

We need to:

- Map potential counterexample derivations in $T_0 \uplus T_1$ back into a $T_0$ derivation (its “skeleton”)

Martin Churchill

Modular Semantics for Open Transition Rules with Negative Premises
Modularity for Negative Steps

We need to:

- Map potential counterexample derivations in $T_0 \uplus T_1$ back into a $T_0$ derivation (its “skeleton”)

![Diagram](image-url)
Modularity for well-supported proofs

Theorem (Modularity)

Suppose $T_0 \uplus T_1$ is a disjoint extension of $T_0$ and let $\pi$ be a well-supported proof for $\frac{H}{\phi}$ in $T_0$.

Then $\pi$ is a well-supported proof for $\frac{H}{\phi}$ in $T_0 \uplus T_1$. 
Conservativity
Source dependence

Now seek to show: in disjoint extensions, no new proofs of old formulae.
Requires source dependence:

each variable in a rule can be traced back to a variable in the source of the conclusion (via transitions in the premise)

Ok: $x \xrightarrow{l} x'$

$\vdash x; y \xrightarrow{l} x'; y$
Source dependence

Now seek to show: in disjoint extensions, no new proofs of old formulae.
Requires *source dependence*:

Each variable in a rule can be traced back to a variable in the source of the conclusion (via transitions in the premise)

\[
\frac{x \xrightarrow{l} x'}{x; y \xrightarrow{l} x'; y}
\]

**Example**

Consider a TSS $\frac{x \xrightarrow{b} 1}{0 \xrightarrow{a} 1}$. Then $0 \xrightarrow{a} 1$ not provable.

Extend by constant 2 with $2 \xrightarrow{b} 1$. Then $0 \xrightarrow{a} 1$ is provable.
Theorem (Conservativeness for Disjoint Extensions)

Let $T_0 \uplus T_1$ be a disjoint extension of $T_0$, where $T_0$ is source-dependent, and let $\phi \in T_0$. Let $\pi$ be a well-supported proof of $\frac{H}{\phi}$ in $T_0 \uplus T_1$. Then $\pi$ is a well-supported proof of $\frac{H}{\phi}$ in $T_0$.

Proof: induction using “source dependence measure” for positive steps. For negative steps, uses modularity result to move counterexamples from $T_0$ to $T_0 \uplus T_1$. 

Soundness over Closed-instance Semantics

**Theorem**

For closed $\phi$, if $\overline{\phi}$ has a well-supported proof then it has a ground well-supported proof.

**Proof:** Follows from the fact that $\overline{\phi}$ has a closed well-supported proof (instantiation closure).
Conservativity over Closed-instance Semantics

Needs source dependence:

Example

Consider TSS $T$ with deduction rule $\frac{x \ b \rightarrow 1}{0 \ a \rightarrow 1}$.

Then $0 \ a \rightarrow$ has a ground well-supported proof (no valid derivations concluding $0 \ a \rightarrow$).

But no well-supported proof in $T$. 
Theorem

In a source dependent system and closed $\phi$, if $\bar{\phi}$ has a ground well-supported proof then it has a well-supported proof.

Proof: Follows from the fact that each derivation of $s \xrightarrow{l} s'$ for closed $s$ is closed.
Conclusions
Contribution

Our notion satisfies:

- Consistency (\(s \xrightarrow{I} s'\) and \(s \nrightarrow\) can’t both be provable)
- Instantiation closure (if \(\bar{s}\) is provable then so is \(\sigma(s)\))
- Modularity (under disjoint extensions, old proofs remain valid)

Assuming source dependent rules:

- Agrees with original notion on closed terms
- Conservativity (under disjoint extensions, no new proofs of old formulae)
Open Algebraic Laws

Consider an algebraic law, like

\[(x; y); z \sim x; (y; z)\]

As the language is (disjointly) extended, the domain of quantification \((x, y, z)\) increases. Ideal:

- we prove such laws in the “minimal subsystem” containing just the rules for \(\sim\);
- guaranteed to hold in any extension \(=\) any system containing this notion of \(\sim\);
Fh-bisimulation

To prove such laws, we need to consider a notion of bisimulation for open terms satisfying this modularity property.

fh-bisimulation is such a notion:

- if \( s R t \) and \( \frac{H}{s \xrightarrow{l} s'} \) then \( \frac{H}{t \xrightarrow{l} t'} \) with \( s' R t' \)

(usual ‘step’ condition, but under arbitrary hypotheses on variables.)

This notion is modular – preserved by disjoint extensions. [Mosses et al.(2010)Mosses, Mousavi, and Reniers]
...with negative premises

- The work here can be used to adapt fh-bisimulation to the negative setting.
- Modularity of the underlying well-supported proofs leads to modularity for the proved equations.
- Another key issue: compositionality (bisimulation as a congruence, via rule formats) [Mousavi et al.(2007)Mousavi, Reniers, and Groote]
PLanCompS vision

- A growing repository of fundamental constructs (like ;) specified independently
- Laws about such constructs can be proved once and for all
  - e.g. associativity/commutativity/unit laws
- Formal semantics can be given in an accessible manner by translation into funcons
  - Tool support – e.g. running programs
- Computational effects via the mechanics of Modular SOS [Mosses(2004), Churchill and Mosses(2013)]

www.plancomps.org
Conclusions

We:

- Extended well-supported proofs to open transition rules
- Proved consistency, instantiation, modularity, conservativity results

Further directions:

- Use these results to support modularity of equational laws
- Consider compositionality of fh-bisimulation based on these notions
- ... 

Thank You.
Luca Aceto and Anna Ingólfsdóttir.

On the expressibility of priority.

Martin Churchill and Peter D. Mosses.

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Robustness of equations under operational extensions.

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SOS rule formats and meta-theory: 20 years after.