Appendix A Proofs of Congruence Results

Lemma 20 (Main text: Lemma 8). *Let r be a pattern and σ a substitution with $\mathsf{dom}(\sigma) = \mathsf{vars}(r)$. Let R' denote the reflexive congruence closure of a vc-bisimulation R and suppose $r[\sigma]R't$. Then exists τ with $\mathsf{dom}(\tau) = \mathsf{vars}(r)$ such that $t \Rightarrow r[\tau]$ and for each x, $\sigma(x)R'\tau(x)$.*

Proof. We proceed by induction on r. If r is a variable, we can set $\tau = \{r \mapsto t\}$ and we are done by reflexivity of $\Rightarrow$. Otherwise, $r = v(r_1, \ldots, r_n)$ for value constructor v and patterns $r_1, \ldots, r_n$. Then $r[\sigma] = v(r_1[\sigma], \ldots, r_n[\sigma])R't$.

If this holds for reflexivity reasons, we can set $\tau = \sigma$ and are done by reflexivity of R' and $\Rightarrow$. If this holds for congruence reasons, we must have $t = v(t_1, \ldots, t_n)$ with $r_i[\sigma] = r_i[\sigma|_{\mathsf{vars}(r_i)}]R't_i$ for $1 \leq i \leq n$. By applying the inductive hypothesis to each, we can find τ_i with $t_i \Rightarrow r_i[\tau_i]$ and $\sigma(x)R\tau_i(x)$ for each x. Let $\tau = \biguplus \tau_i$. By applying the precongruence rule, $t = v(t_1, \ldots, t_n) \Rightarrow v(r_1, \ldots, r_n)[\tau] = r[\tau]$ with $\sigma(x)R'\tau(x)$ for each x, as required.

If $r[\sigma] = v(r_1[\sigma], \ldots, r_n[\sigma])Rt$, then since R is a vc-bisimulation we must have $t \Rightarrow v(t_1, \ldots, t_n)$ with $r_i[\sigma] = r_i[\sigma_{\mathsf{vars}(r_i)}]R't_i$ for $1 \leq i \leq n$. By applying the inductive hypothesis to each, we find τ_i with $t_i \Rightarrow r_i[\tau_i]$ and $\sigma(x)R\tau_i(x)$ for each x. Let $\tau = \biguplus \tau_i$. By applying the precongruence rule, $v(t_1, \ldots, t_n) \Rightarrow v(r_1, \ldots, r_n)[\tau] = r[\tau]$. By applying the transitivity rule, we see that $t \Rightarrow r[\tau]$ with $\sigma(x)R'\tau(x)$ for each x, as required. □

Theorem 21 (Main text: Theorem 9). *If all rules in a value-computation transition system are defined in the value-added tyft format and well-founded, then vc-bisimilarity in that system is a congruence.*

Proof. We first remove each value variable appearing in the rules. If a value variable v appears in a rule R, we may remove it by adding an additional rule for each value constructor f, replacing v for $f(x_1, \ldots, x_n)$ where $n = \mathsf{ar}(f)$ and x_i are fresh variables. The result is still in the value-added tyft format.

Let R be a vc-bisimulation, and let R' denote the reflexive congruence closure of R. We will show that R' is also a vc-bisimulation, and since R' contains R we can conclude that vc-bisimilarity is a congruence.

To show that R' is a bisimulation, we must show the three conditions in Definition 5. The third condition follows immediately from the above lemma: If $s = v(s_1, \ldots, s_n)R't$ then $s = r[\sigma]$ where $r = v(x_1, \ldots, x_n)$ and $\sigma = \{x_i \mapsto s_n\}$. By the lemma, $t \Rightarrow r[\tau]$ with $\tau = \{x_i \mapsto t_i\}$ and $s_iR't_i$. Then $t = v(t_1, \ldots, t_n)$ as required.

We next prove conditions (1) and (2) simultaneously, showing that $s \rightsquigarrow s'$ and $sR't$ implies exists t' with $t \rightsquigarrow t'$ with $s'R't'$ for any $\rightsquigarrow$ of the form $\Rightarrow$ or $\xrightarrow{a}$. We write $\rightsquigarrow$, possibly with subscripts, for variables ranging over such arrows. Note that by using either saturation or transitivity, the following is always admissible:

$$\frac{s \Rightarrow s_1 \qquad s_1 \rightsquigarrow s_2 \qquad s_2 \Rightarrow s'}{s \rightsquigarrow s'}$$

We will refer to this rule as ‘generalised saturation’.

Suppose $s \rightsquigarrow s'$ with $sR't$. We proceed by induction on the proof that $s \rightsquigarrow s'$. We consider case analysis on $sR't$. If sRt then the step condition holds using the fact that R is a vc-bisimulation. If $s = t$ then the step condition is trivial. In the final case, $s = f(s_1, \ldots, s_n)$ and $t = f(t_1, \ldots, t_n)$ with $s_i R' t_i$. Let $s \rightsquigarrow s'$ be witnessed by a proof γ with immediate children $\{\gamma_i : i \in I\}$. Let $\mathcal{R}$ denote the final rule used in γ under substitution σ.

Now $\mathcal{R}$ is either one of the built in rules or a value-added tyft rule.

Tyft rule: Suppose $\mathcal{R}$ concludes $r \rightsquigarrow r'$ from $\{u_i \rightsquigarrow_i u'_i : i \in I\}$ with $r[\sigma] = s$, $r'[\sigma] = s'$ and with γ_i a proof of $u_i[\sigma] \rightsquigarrow_i u'_i[\sigma]$. Since $\mathcal{R}$ is in the value-added tyft format, we must have $r = f(r_1, \ldots, r_n)$ with $r_j[\sigma] = s_j$ where each r_j is a pattern.

Now, $s_j = r_j[\sigma|_{\mathsf{vars}(r_j)}] R' t_j$ for $1 \leq j \leq n$. We apply the lemma for each such j and let τ denote the disjoint union of the resulting substitutions. Thus, $\sigma(x) R \tau(x)$ for each x in $\mathsf{vars}(r)$, and $t_j \Rightarrow r_j[\tau]$. By applying the precongruence rule at f, $t \Rightarrow r[\tau]$.

Well-foundedness ensures that variables in u'_i do not appear in u_j,u'_j for $j < i$. We will now show that for each $i \in I$ there exists a substitution τ_i such that $\tau_i \supseteq \tau$; for each $j < i$, $\tau_i \supseteq \tau_j$; $u_i[\tau_i] \rightsquigarrow_i u'_i[\tau_i]$; and for each $x \in \mathsf{dom}(\tau_i)$, $\sigma(x) R \tau_i(x)$.

We proceed by (ordinal) induction on i. Let $\tau_i^1 = \tau \cup \bigcup_{j<i} \tau_j$. Let $\tau_i^2 = \tau_i^1 \uplus \sigma|_{\mathsf{vars}(u_i) - \mathsf{dom}(\tau_i^1)}$. Now, we know that $s_i = u_i[\sigma]$. By the (inner) inductive hypothesis we see that for each $x \in \mathsf{dom}(\tau_i^2)$, $\tau_i^2(x) R' \sigma(x)$. Let $t_i = u_i[\tau_i^2]$. Then, since R' is a congruence, $s_i = u_i[\sigma] R' u_i[\tau_i^2] = t_i$. Since $s_i \rightsquigarrow_i s'_i$ we may apply the (outer) inductive hypothesis to see that there exists t'_i such that $t_i \rightsquigarrow_i t'_i$ with $s'_i R' t'_i$. But also note that $s'_i = u'_i[\sigma]$ and u'_i is a pattern. By applying the lemma, we can find τ_i^3 such that $t'_i \Rightarrow u'_i[\tau_i^3]$ with $\sigma(x) R' \tau_i^3(x)$ for each $x \in \mathsf{dom}(\tau_i^3) = \mathsf{vars}(u'_i)$. Let $\tau_i = \tau_i^2 \uplus \tau_i^3$. Then $u_i[\tau_i] = u_i[\tau_i^2] = t_i \rightsquigarrow_i t'_i \Rightarrow u'_i[\tau_i]$. By applying generalised saturation, we see that $u_i[\tau_i] \rightsquigarrow_i u'_i[\tau_i]$. Since $\tau_i \supseteq \tau_j$ for $j < i$ and $\forall x \in \mathsf{dom}(\tau_i)$, $\sigma(x) R' \tau_i(x)$, we are done.

Now, let $\tau' = \bigcup_{i \in I} \tau_i$. Then $\forall x \in \mathsf{dom}(\tau')$, $\sigma(x) R' \tau'(x)$. Also for each i, $u_i[\tau'] \rightsquigarrow_i u'_i[\tau']$. We may thus apply the rule $\mathcal{R}$ and conclude that $r[\tau'] \rightsquigarrow r'[\tau']$. Since $t \Rightarrow r[\tau']$ we may apply generalised saturation and conclude that $t \rightsquigarrow t'$ where $t' = r'[\tau']$. We only need to show that $s' R t'$. Well $s' = r'[\sigma]$ and $t' = r'[\tau']$, and $\sigma(x) R' \tau'(x)$ for each x. Since R' is a congruence, it follows that $s' R' t'$.

Built-in rule: If $\mathcal{R}$ is the saturation rule then $\rightsquigarrow = \xrightarrow{a}$ and the premises of γ are $s \Rightarrow s_1$, $s_1 \xrightarrow{a} s_2$ and $s_2 \Rightarrow s'$. Since $sR't$ and $s \Rightarrow s_1$, by inductive hypothesis $t \Rightarrow t_1$ with $s_1 R' t_1$. Then by inductive hypothesis $t_1 \xrightarrow{a} t_2$ with $s_2 R' t_2$. Finally, by inductive hypothesis $t_2 \Rightarrow t'$ with $s' R' t'$. By applying the saturation rule, $t \xrightarrow{a} t'$ with $s' R' t'$, as required. The transitivity rule for $\Rightarrow$ is entirely similar.

If $\mathcal{R}$ is reflexivity for $\Rightarrow$, then $s' = s$ and we can simply take $t' = t$.

If $\mathcal{R}$ is the precongruence rule for $\Rightarrow$ then we must have $s_i \Rightarrow s'_i$ with $s' = f(s'_1, \ldots, s'_n)$. Since $s_i R t_i$ we may apply the inductive hypothesis to see that $t_i \Rightarrow t'_i$ with $s'_i R' t'_i$. By applying the precongruence rule we see that $t \Rightarrow$

$t' = f(t'_1, \ldots, t'_1)$. As R' is a precongruence, we find that $s'R't'$, and the proof is complete. □

We will now move to the MSOS setting. We use variables such as $\rightsquigarrow$ to range over arrows $\xrightarrow{L}$ and $\Rightarrow$. Define $\mathsf{reads}(\xrightarrow{L}) = \mathsf{reads}(L)$, $\mathsf{reads}(\Rightarrow) = \emptyset$, $\mathsf{writes}(\xrightarrow{L}) = \mathsf{writes}(L)$ and $\mathsf{writes}(\Rightarrow) = \emptyset$.

Proposition 22 (Main text: Proposition 16). *Each well-founded MSOS tyft system is equivalent to one in the explicit MSOS tyft format.*

Proof. Given an MSOS transition system T over label profile $\mathcal{L}$ we first produce an equivalent set of rules removing all uses of label variables, exhibiting all information flow in labels explicitly. We modify each rule in T. First, let I denote the elements in $\mathsf{reads}(\mathcal{L}) \uplus \mathsf{writes}(\mathcal{L})$ not mentioned explicitly in that rule. Then:

- A variable ranging over arbitrary labels (e.g. '…') and not appearing in a composition is replaced by $\{l = x_l\}$ for fresh variables x_l for each $l \in I$.
- If the conclusion label ends with the empty composition ('−') it is replaced by $\{l = s_l\}$ for $l \in I$, where:
  - For $\mathbf{x} \in \mathcal{L}_{RO}$, $s_\mathbf{x}$ is a fresh variable.
  - For $\mathbf{x} \in \mathcal{L}_{RW}$, $s_\mathbf{x}$ and $s_{\mathbf{x}'}$ are the same fresh variable.
  - For $\mathbf{x} \in \mathcal{L}_{WO}$, $s_{\mathbf{x}'} = \iota_\mathbf{x}$.
- If the conclusion label ends with label variable composition $X_n \circ \ldots \circ X_1$ then we replace each X_j by $\{l = s_{l,j}\}$ and $X_n \circ \ldots \circ X_1$ by $\{l = s_l\}$ for $l \in I$ where:
  - For $\mathbf{x} \in \mathcal{L}_{RO}$, $s_\mathbf{x}$ and each $s_{\mathbf{x},j}$ are the same fresh variable.
  - For $\mathbf{x} \in \mathcal{L}_{RW}$, $s_\mathbf{x}$ and each $s_{\mathbf{x}',j}$ are fresh variables, $s_{\mathbf{x},1} = s_\mathbf{x}$, $s_{\mathbf{x},j+1} = s_{\mathbf{x}',j}$ and $s_{\mathbf{x}'} = s_{\mathbf{x}',n}$.
  - For $\mathbf{x} \in \mathcal{L}_{WO}$, each $s_{\mathbf{x}',j}$ is a fresh variable and $s_{\mathbf{x}'} = t_n$ where $t_0 = s_{\mathbf{x}',0}$ and $t_{j+1} = \odot_\mathbf{x}(t_j, s_{\mathbf{x}',j+1})$.

The resulting system explicitly expresses the underlying mechanics of unobservability and composition, and does so in the MSOS tyft format. □

Proposition 23 (Main text: Proposition 17). *Consider a MSOS specification in explicit MSOS tyft format. Let R be an MSOS bisimulation over the generated transition system and let R' denote the reflexive transitive congruence closure of R. Suppose $sR't$. Then:*

1. *If $s = r[\sigma]$ with $\mathsf{dom}(\sigma) = \mathsf{vars}(r)$ and r is a pattern then exists τ with $\mathsf{dom}(\tau) = \mathsf{vars}(r)$ such that $t \Rightarrow r[\tau]$ with $\sigma(x)R'\tau(x)$ for each $x \in \mathsf{vars}(r)$.*
2. *If $s \rightsquigarrow s'$ and $\mathsf{reads}(\rightsquigarrow)R'trs$ then exists $\rightsquigarrow'$,t' such that $t \rightsquigarrow' t'$, $\mathsf{reads}(\rightsquigarrow') = trs$, $\mathsf{writes}(\rightsquigarrow')R'\mathsf{writes}(\rightsquigarrow')$ and $\rightsquigarrow' = \Rightarrow$ iff $\rightsquigarrow = \Rightarrow$.*

Proof. We first remove each value variable appearing in the rules. If a value variable v appears in a rule R, we may remove it by adding an additional rule for each value constructor f, replacing v for $f(x_1, \ldots, x_n)$ where $n = \mathsf{ar}(f)$ and x_i are fresh variables. The result is still in the MSOS tyft format.

We then proceed by simultaneous induction on R'. We first show condition **1**.

Reflexivity. If $r[\sigma] = t$ then we can take $\tau = \sigma$ and the result follows by reflexivity of $\Rightarrow$ and R'.

Transitivity. If $r[\sigma]R'sR't$ then we may apply inductive hypothesis 1. to construct ρ with $\mathsf{dom}(\rho) = \mathsf{dom}(\sigma) = \mathsf{vars}(r)$ with $s \Rightarrow r[\rho]$ and for each x, $\sigma(x)R'\rho(x)$. Since $sR't$ and $s \Rightarrow r[\rho]$ by applying inductive hypothesis 2. it follows that $t \Rightarrow t'$ for some $r[\rho]R't'$. By inductive hypothesis 1., exists τ with $\mathsf{dom}(\tau) = \mathsf{vars}(r)$ and $t' \Rightarrow r[\tau]$ and for each x $\rho(x)R'\tau(x)$. By applying the transitivity rule for $\Rightarrow$ we see that $t \Rightarrow r[\tau]$. Finally, for each $x \in \mathsf{vars}(r)$, $\sigma(x)R'\rho(x)R'\tau(x)$ and since R' is transitive, $\sigma(x)R'\tau(x)$ as required.

Congruence. If r is a variable, we can set $\tau = \{r \mapsto t\}$ and we are done by reflexivity of $\Rightarrow$. Otherwise, $r = v(r_1, \ldots, r_n)$ for value constructor v and patterns $r_1, \ldots, r_n$. Then $r[\sigma] = v(r_1[\sigma], \ldots, r_n[\sigma])R't$. Since $sR't$ for congruence reasons, we must have $t = v(t_1, \ldots, t_n)$ with $r_i[\sigma] = r_i[\sigma|_{\mathsf{vars}(r_i)}]R't_i$ for $1 \leq i \leq n$. By applying inductive hypothesis 1. to each, we see that $t_i \Rightarrow r_i[\tau_i]$ with $\sigma(x)R'\tau_i(x)$ for each x. Let $\tau = \biguplus \tau_i$. By applying the precongruence rule, $t = v(t_1, \ldots, t_n) \Rightarrow v(r_1, \ldots, r_n)[\tau] = r[\tau]$ with $\sigma(x)R'\tau(x)$ for each x, as required.

Base relation. If r is a variable, we can set $\tau = \{r \mapsto t\}$ and we are done by reflexivity of $\Rightarrow$. Otherwise, if $r[\sigma] = v(r_1[\sigma], \ldots, r_n[\sigma])Rt$, then since R is a MSOS bisimulation we must have $t \Rightarrow v(t_1, \ldots, t_n)$ with $r_i[\sigma] = r_i[\sigma_{\mathsf{vars}(r_i)}]Rt_i$ for $1 \leq i \leq n$. By applying inductive hypothesis 1. to each, we see that $t_i \Rightarrow r_i[\tau_i]$ with $\sigma(x)R'\tau_i(x)$ for each x. Let $\tau = \biguplus \tau_i$. By applying the precongruence rule, $v(t_1, \ldots, t_n) \Rightarrow v(r_1, \ldots, r_n)[\tau] = r[\tau]$. By applying the transitivity rule, we see that $t \Rightarrow r[\tau]$ with $\sigma(x)R'\tau(x)$ for each x, as required.

We next show condition **2**.

Congruence. We now consider the case that $s = f(s_1, \ldots, s_n)$ and $t = f(t_1, \ldots, t_n)$ with $s_iR't_i$. We proceed by a 2nd level induction on the proof that $s \rightsquigarrow s'$. Let $\mathcal{R}$ denote the last rule used in this proof. We perform case analysis on the nature of $\mathcal{R}$.

$\mathcal{R}$ is a MSOS tyft rule. Suppose $\mathcal{R}$ concludes $r \rightsquigarrow^{\mathcal{R}} r'$ from $\{r_i \rightsquigarrow^{\mathcal{R}}_i r'_i : i \in I\}$ with substitution σ. Let $rs = \mathsf{dom}(\mathsf{reads}(\rightsquigarrow^{\mathcal{R}}))$ and $rs_x = \mathsf{reads}(\rightsquigarrow^{\mathcal{R}})(x)$. Then for each x, $rs_x[\sigma]R'trs_x$. Since rs_x is a pattern, we may apply condition 1. to construct τ^1 with $trs_x \Rightarrow rs_x[\tau^1]$ and for each $x \in \mathsf{dom}(\tau^1)$, $\sigma(x)R'\tau(x^1)$. Also $r[\sigma]Rt$ and we may construct τ^2 similarly, with $t \Rightarrow r[\tau^2]$, $\mathsf{dom}(\tau^2) = \mathsf{vars}(r)$, and for all $x \in \mathsf{vars}(r)$, $\sigma(x)R'\tau^2(x)$. Let $\tau = \tau^1 \uplus \tau^2$.

We now construct a sequence of substitutions τ_i. For each i we will show that: $\tau_i \supseteq \tau$, $\tau_i \supseteq \tau_j$ for $j < i$, $(r_i \rightsquigarrow^{\mathcal{R}}_i r'_i)[\tau_i]$ and for all x in $\mathsf{dom}(\tau_i)$, $\sigma(x)R'\tau_i(x)$. We proceed by a 3rd level (ordinal) induction on i.

Let $\tau^1_i = \tau \cup \bigcup_{j<i} \tau_j$. By 3rd level induction, for all x in $\mathsf{dom}(\tau^1_i)$, $\sigma(x)R'\tau^1_i(x)$. Since R' is a congruence $r_i[\sigma]R'r_i[\tau^1_i(x)]$. Also, $\mathsf{reads}(\rightsquigarrow^{\mathcal{R}}_i)[\sigma]R'\mathsf{reads}(\rightsquigarrow^{\mathcal{R}}_i)[\tau^1_i]$. Also, $(r_i \rightsquigarrow^{\mathcal{R}}_i r'_i)[\sigma]$. By inductive hypothesis 2., we may find t'_i and $\rightsquigarrow'_i$ such that $r_i[\tau^1_i(x)] \rightsquigarrow'_i t'_i$ with $r'_i[\sigma]R't'_i$, $\mathsf{reads}(\rightsquigarrow'_i) = \mathsf{reads}(\rightsquigarrow^{\mathcal{R}}_i)[\tau^1_i]$ and $\mathsf{writes}(\rightsquigarrow^{\mathcal{R}}_i)[\sigma]R'$ $\mathsf{writes}(\rightsquigarrow'_i)$. For each x in $\mathsf{dom}(\mathsf{writes}(\rightsquigarrow_i))$, $\mathsf{writes}(\rightsquigarrow^{\mathcal{R}}_i)(x)[\sigma]R'\mathsf{writes}(\rightsquigarrow'_i)(x)$. But $\mathsf{writes}(\rightsquigarrow^{\mathcal{R}}_i)(x)$ is a pattern. We may apply condition 1. to find that $\mathsf{writes}(\rightsquigarrow'_i)(x) \Rightarrow$

$\mathsf{writes}(\rightsquigarrow_i^{\mathcal{R}})(x)[\tau_i^{2,x}]$. Also, since $r'_i[\sigma]R't'_i$ we may apply condition 1 to find τ_i^2 with $t'_i \Rightarrow r'_i[\tau_i^2]$. Let τ_i denote the disjoint union of τ_i^1, τ_i^2 and each $\tau_i^{2,x}$. Then $\mathsf{writes}(\rightsquigarrow'_i)(x) \Rightarrow \mathsf{writes}(\rightsquigarrow_i^{\mathcal{R}})(x)[\tau_i]$ and $t'_i \Rightarrow r'_i[\tau_i]$. By applying the saturation rule (for t'_i and each $\mathsf{writes}(\rightsquigarrow'_i)(x)$) we see that $(r_i \rightsquigarrow_i^{\mathcal{R}} r'_i)[\tau_i]$. Also, for each x in $\mathsf{dom}(\tau_i)$, $\sigma_i(x)R'\tau_i(x)$, as required.

Let $\tau' = \bigcup_{i \in I} \tau_i$. We can apply the rule $\mathcal{R}$ to each $(r_i \rightsquigarrow_i^{\mathcal{R}} r'_i)[\tau']$ to conclude that $(r \rightsquigarrow^{\mathcal{R}} r')[\tau']$. Let $t' = r'[\tau']$. Then $t \Rightarrow r[\tau] = r[\tau'] \rightsquigarrow^{\mathcal{R}} [\tau']t'$. Let $\rightsquigarrow' = \rightsquigarrow^{\mathcal{R}} [\tau']$. Then by saturation $t \rightsquigarrow' t'$. Also $s'R't'$ since $t' = r'[\tau']$ and $s' = r'[\sigma]$ and for each x, $\sigma(x)R'\tau'(x)$. Also $\mathsf{writes}(\rightsquigarrow) = \mathsf{writes}(\rightsquigarrow^{\mathcal{R}} [\sigma])R'\mathsf{writes}(\rightsquigarrow^{\mathcal{R}} [\tau']) = \mathsf{writes}(\rightsquigarrow')$ and similarly $\mathsf{reads}(\rightsquigarrow) = \mathsf{reads}(\rightsquigarrow')$.

$\mathcal{R}$ is a saturation rule. If $\mathcal{R}$ is a source-target saturation rule and $s \Rightarrow s_1 \rightsquigarrow s_2 \Rightarrow s'$ with $sR't$ we may apply the inductive hypothesis to each case sequentially, constructing t_1 and t_2 such that $t \Rightarrow t_1 \rightsquigarrow' t_2 \Rightarrow t'$ with $s_1R't_1$, $s_2R't_2$, $s'R't'$ and $\mathsf{writes}(\rightsquigarrow)R'\mathsf{writes}(\rightsquigarrow')$ and $\mathsf{reads}(\rightsquigarrow') = trs$. By applying the saturation rule, we can conclude that $t \rightsquigarrow' t'$ as required.

If $\mathcal{R}$ is a read-saturation rule under component x, then $\mathsf{reads}(\rightsquigarrow)(x) \Rightarrow \mathsf{reads}(\rightsquigarrow_1)(x)$ and $s \rightsquigarrow_1 s'$. Since $\mathsf{reads}(\rightsquigarrow)(x)R'trs(x)$ by inductive hypothesis 2. $trs(x) \Rightarrow p$ with $\mathsf{reads}(\rightsquigarrow_1)(x)R'p$. Let $trs' = trs[x \mapsto p]$. Then $trs'R'\mathsf{reads}(\rightsquigarrow_1)$. Since $sR't$ by inductive hypothesis, exists $\rightsquigarrow'_1$ with $t \rightsquigarrow'_1 t'$ with $s'Rt'$ and $\mathsf{reads}(\rightsquigarrow'_1) = trs'$ and $\mathsf{writes}(\rightsquigarrow_1)R'\mathsf{writes}(\rightsquigarrow'_1)$. By applying the read-saturation rule, we find that $s \rightsquigarrow_1 t'$ with $s'R't'$, $\mathsf{reads}(\rightsquigarrow_1) = trs$ and $\mathsf{writes}(\rightsquigarrow_1) = \mathsf{writes}(\rightsquigarrow'_1)R'\mathsf{writes}(\rightsquigarrow_1) = \mathsf{writes}(\rightsquigarrow)$ as required.

If $\mathcal{R}$ is a write-saturation rule under x, then $s \rightsquigarrow_1 s'$ and $\mathsf{writes}(\rightsquigarrow_1)(x) \Rightarrow p$ and $\rightsquigarrow = \rightsquigarrow_1[x \mapsto p]$. Since $\mathsf{writes}(\rightsquigarrow_1)(x) \Rightarrow \mathsf{writes}(\rightsquigarrow_1)(x)$, by inductive hypothesis 2. we must have $pR'\mathsf{writes}(\rightsquigarrow_1)(x)$. By inductive hypothesis 2. again since sRt, exists $\rightsquigarrow'_1$ with $t \rightsquigarrow'_1 t'$ with $s'Rt'$, $\mathsf{reads}(\rightsquigarrow'_1) = trs$ and $\mathsf{writes}(\rightsquigarrow_1)R'$ $\mathsf{writes}(\rightsquigarrow'_1)$. By applying the write saturation rule, $t \rightsquigarrow' t'$ where $\rightsquigarrow' = \rightsquigarrow'_1 [x \mapsto p]$. Now $\mathsf{reads}(\rightsquigarrow') = trs$ and $\mathsf{writes}(\rightsquigarrow)R'\mathsf{writes}(\rightsquigarrow')$, since: for each $y \neq x$ we have $\mathsf{writes}(\rightsquigarrow)(y) = \mathsf{writes}(\rightsquigarrow_1)(y)R\mathsf{writes}(\rightsquigarrow'_1)(y) = \mathsf{writes}(\rightsquigarrow')(y)$; for x we have $\mathsf{writes}(\rightsquigarrow)(x) = pR'\mathsf{writes}(\rightsquigarrow')(x)$.

$\mathcal{R}$ is a built-in $\Rightarrow$ rule. The cases of reflexivity, transitivity and precongruence follow simply by induction as in Theorem 9, nothing that in such cases both $\rightsquigarrow$ and $\rightsquigarrow'$ are $\Rightarrow$ and so no labels are involved.

Reflexivity. Suppose $s = t$. If $\rightsquigarrow = \Rightarrow$ the case is trivial as $trs = \emptyset$ and we can take $t' = s'$. Otherwise, we proceed by a 2nd level induction on the proof that $s \rightsquigarrow s'$. Let $\mathcal{R}$ denote the last rule used in this proof.

$\mathcal{R}$ is a tyft rule. Suppose $\mathcal{R}$ concludes $r \rightsquigarrow^{\mathcal{R}} r'$ from $\{r_i \rightsquigarrow_i^{\mathcal{R}} r'_i : i \in I\}$ with substitution σ. Let $rs = \mathsf{dom}(\mathsf{reads}(\rightsquigarrow^{\mathcal{R}}))$ and $rs_x = \mathsf{reads}(\rightsquigarrow^{\mathcal{R}})(x)$. Then for each x, $rs_x[\sigma]R'trs_x$. We may apply condition 1. to construct τ^1 with $trs_x \Rightarrow rs_x[\tau^1]$ and for each $x \in \mathsf{dom}(\tau)$, $\sigma(x)R'\tau^1(x)$. Let $\tau = \tau^1 \cup \sigma|_{\mathsf{vars}(r)}$. By reflexivity of R', for each $x \in \mathsf{dom}(\tau)$, $\sigma(x)R'\tau(x)$.

We now construct a sequence of substitutions τ_i. For each i we will show that: $\tau_i \supseteq \tau$, $\tau_i \supseteq \tau_j$ for $j < i$, $(r_i \rightsquigarrow_i^{\mathcal{R}} r'_i)[\tau_i]$ and for all x in $\mathsf{dom}(\tau_i)$, $\sigma(x)R'\tau_i(x)$. We proceed by a 3rd level (ordinal) induction on i, and the proof is exactly as in the congruence case above.

Let $\tau' = \bigcup_{i \in I} \tau_i$. We can apply the rule $\mathcal{R}$ to each $(r_i \rightsquigarrow_i^{\mathcal{R}} r_i')[\tau']$ to conclude that $(r \rightsquigarrow^{\mathcal{R}} r')[\tau']$. Let $t' = r'[\tau']$ and $\rightsquigarrow' = \rightsquigarrow^{\mathcal{R}} [\tau']$. Then $s = t \rightsquigarrow' t'$. Also $s'R't'$ since $t' = r'[\tau']$ and $s' = r'[\sigma]$ and for each x, $\sigma(x)R'\tau'(x)$. Also, $\mathsf{writes}(\rightsquigarrow)R'\mathsf{writes}(\rightsquigarrow')$ and $\mathsf{reads}(\rightsquigarrow) = \mathsf{reads}(\rightsquigarrow')$.

$\mathcal{R}$ is a saturation rule. The saturation rules follow exactly as in the congruence case (the proof of this particular subcase didn't actually use congruence) in the degenerate case that $s = t$.

Transitivity. Let $s \rightsquigarrow s'$ and $sR'rR't$ and $\mathsf{reads}(\rightsquigarrow)R'trs$. Then by inductive hypothesis $r \rightsquigarrow' r'$ with $\mathsf{reads}(\rightsquigarrow') = trs$ and $\mathsf{writes}(\rightsquigarrow)R'\mathsf{writes}(\rightsquigarrow')$. By inductive hypothesis again $t \rightsquigarrow'' t'$ with $\mathsf{reads}(\rightsquigarrow'') = trs$ and $\mathsf{writes}(\rightsquigarrow')R'\mathsf{writes}(\rightsquigarrow'')$ and $r'R't'$. Since R' is transitive, $s'R't'$ and $\mathsf{writes}(\rightsquigarrow)R'\mathsf{writes}(\rightsquigarrow'')$, and we are done.

Base relation. Suppose then that sRt and $s \rightsquigarrow s'$. Using the reflexivity case above, since $sR's$, we can find $\rightsquigarrow'$ and s'' such that $s \rightsquigarrow' s''$, $s'R's''$, $\mathsf{reads}(\rightsquigarrow') = trs$ and $\mathsf{writes}(\rightsquigarrow)R'\mathsf{writes}(\rightsquigarrow')$. Then, since sRt and R is an MSOS bisimulation, we find that $t \rightsquigarrow'' t'$ with $s''Rt'$, $\mathsf{reads}(\rightsquigarrow'') = \mathsf{reads}(\rightsquigarrow') = trs$ and $\mathsf{writes}(\rightsquigarrow')R\mathsf{writes}(\rightsquigarrow'')$. Since $s'R's''R't'$ and R' is transitive, we have $s'R't'$. Since $\mathsf{writes}(\rightsquigarrow')R'\mathsf{writes}(\rightsquigarrow'')$ and $\mathsf{writes}(\rightsquigarrow'')R\mathsf{writes}(\rightsquigarrow')$ we find that $\mathsf{writes}(\rightsquigarrow')$ R' $\mathsf{writes}(\rightsquigarrow)$, as required. □

Theorem 24 (Main text: Theorem 18). *Consider an MSOS system T in the well-founded MSOS tyft format. Let R be an MSOS bisimulation and R' denote the reflexive transitive congruence closure of R. Then R' is an MSOS bisimulation.*

Proof. We first convert T into equivalent T' which is an explicit MSOS tyft format following Proposition 16. We then show that R' is an MSOS bisimulation by considering the three conditions in turn:

1. Follows by Proposition 17 point 2. taking $trs = \mathsf{reads}(L)$.
2. This condition is precisely Proposition 17 point 3., corresponding to point 2. in the above proof taking $\rightsquigarrow = \rightsquigarrow' = \Rightarrow$.
3. If $s = v(s_1, \ldots, s_n)R't$ then $s = r[\sigma]$ where $r = v(x_1, \ldots, x_n)$ and $\sigma = \{x_i \mapsto s_n\}$. By applying Proposition 17 point 1, $t \Rightarrow r[\tau]$ with $\tau = \{x_i \mapsto t_i\}$ and $s_iR't_i$. Then $t = v(t_1, \ldots, t_n)$ as required. □

Corollary 25. *MSOS-bisimilarity is a congruence for specifications in the well-founded MSOS tyft format.*